J-matrix method and Bargmann potentials

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Abstract

By applying the J-matrix method [1] to neutral particles scattering we have discovered that there is a one-to-one correspondence between the nonlocal separable potential with the Laguerre form factors and a Bargmann potential. Thus this discrete approach to direct and inverse scattering problem can be considered as a tool of the S-matrix rational parametrization. As an application, the Bargmann potentials, phase-equivalent to the np $^1S_0$ Yamaguchi potential [7] and to the np potential from inverse scattering in the J-matrix approach [6] have been obtained.

1 Introduction

In the J-matrix method [1] the initial short-ranged interaction is approximated by the model nonlocal separable potentials with the harmonic oscillator or Laguerre form factors. In the latter case the S-matrix is a rational function of the wave number k. This allows to establish the correspondence between the model potential and the phase-equivalent Bargmann potential [2, 3]. Bargmann's type potentials were used extensively for nuclear systems description (see e. g. [4] and references therein). Recent results see in Ref. [5]. As a rule the model potentials of rather high rank N ($N \geq 10$) are necessary for the J-matrix description for the initial interaction. This makes solving the Marchenko integral equation for the Bargmann potentials difficult. On the other hand, the inverse scattering method within the J-matrix approach [6] enables one to construct the model potentials of low rank. In view of noticed the S-matrix property of the model potential, this inverse scattering method is at once an efficient tool for the Bargmann potentials construction.

In Sec. 2 the S-matrix of the separable potential with the Laguerre form factors was demonstrated to be a rational function of the wave number k. In Sec. 3 for the Bargmann potential Jost function in terms of the results of inversion procedure within the J-matrix approach [6] is given in an explicit form. In Sec. 4 the Bargmann potentials have been obtained, phase-equivalent to the np $^1S_0$ potentials: (i) Yamaguchi of rank 1 potential [7]; (ii) separable potential of rank 4 from inverse scattering in the J-matrix method. In Sec. 5 we summarize our conclusions.

2 The model separable potentials

In the J-matrix approach [1] the short-ranged potential in the partial wave ℓ is approximated by a separable expansion:

$$\widehat{V}^{\ell} = \frac{\hbar^2}{2\mu} \sum_{n,\, n'=0}^{N-1} \left|\overline{\phi}_n^{\ell}\right\rangle V_{n,\,n'} \left\langle \overline{\phi}_{n'}^{\ell}\right|, \tag{1}$$

where the form factors

$$\left|\overline{\phi}_n^{\ell}\right\rangle = \frac{n!}{r\,(n+2\ell+1)!} (2br)^{\ell+1}\, e^{-br} L_n^{2\ell+1}(2br) \tag{2}$$

are bi-orthogonal to the Laguerre basis functions

$$\left|\phi_n^\ell\right\rangle = (2br)^{\ell+1}\; e^{-br} L_n^{2\ell+1}(2br), \tag{3}$$

i. e.

$$\int\limits_0^\infty \phi_n^\ell(r)\,\overline{\phi}_{n'}^\ell(r)\,dr = \delta_{n\,n'}. \tag{4}$$

Here, b is a scaling parameter.

In the neutral particles scattering case the Fredholm determinants $\mathcal{D}^{(\pm)}(k)$ corresponding to the Lippman-Schwinger equation solutions is given by (see e. g. [8]):

$$\mathcal{D}^{(\pm)}(k) \equiv \det(\mathbf{I} - \mathbf{G}^{(0)(\pm)}\mathbf{V}). \tag{5}$$

Here, $[\mathbf{V}]_{n,\,n'} = V_{n,\,n'}$ are the matrix elements of the potential (1); the matrix elements of the free Green's operator $\widehat{G}^{(0)(\pm)}(k) = \left[k^2 \pm \mathrm{i}\varepsilon - \widehat{h}_0\right]^{-1}$ $(\widehat{h}_0 = -\dfrac{d^2}{d\,r^2} + \dfrac{\ell(\ell+1)}{r^2})$ are defined by the expressions [9, 10]:

$$[\mathbf{G}^{(0)(\pm)}]_{n,\,n'} = \langle\overline{\phi}_n^\ell \left|\widehat{G}^{(0)(\pm)}(k)\right| \overline{\phi}_{n'}^\ell\rangle = -\frac{1}{k}\, S_{n_<}^\ell(k) C_{n_>}^{\ell(\pm)}(k) \tag{6}$$

where

$$\begin{aligned} C_n^{\ell(\pm)}(k) &= -\frac{n!\, e^{\mp\mathrm{i}(n+1)\,\theta}}{(n+\ell+1)!\,(2\,\sin\theta)^\ell}\; {}_2F_1(-\ell,\, n+1;\, n+\ell+2;\, e^{\mp 2\mathrm{i}\theta}), \\ S_n^\ell(k) &= \frac{\ell!\,(2\,\sin\theta)^{\ell+1}}{2\,(n+2\ell+1)!}\; \mathcal{C}_n^{\ell+1}(\cos\theta), \\ e^{\mathrm{i}\theta} &= (k+\mathrm{i}b)/(k-\mathrm{i}b), \end{aligned} \tag{7}$$

$\mathcal{C}_n^{\ell+1}(x)$ represent the Gegenbauer polynomials. Reference to Eqs. (6), (7), shows that the Fredholm determinants $\mathcal{D}^{(\pm)}(k)$ (5) are rational functions of the wave number k, furthermore, as shown below, $\mathcal{D}^{(\pm)}(k)$ can be written in the form:

$$\mathcal{D}^{(\pm)}(k) = \frac{\prod\limits_{j=1}^{\mathcal{N}} (k \pm \mathrm{i}a_j)}{(k \pm \mathrm{i}b)^{\mathcal{N}}}, \qquad \mathcal{N} = 2(N+\ell). \tag{8}$$

Notice that since S-matrix of the potential (1) is defined by [8]

$$S(k) = \frac{\mathcal{D}^{(-)}(k)}{\mathcal{D}^{(+)}(k)}, \tag{9}$$

it is also rational function of k:

$$S(k) = \left(\frac{k+\mathrm{i}b}{k-\mathrm{i}b}\right)^{\mathcal{N}} \prod_{j=1}^{\mathcal{N}} \frac{(k-\mathrm{i}a_j)}{(k+\mathrm{i}a_j)}. \tag{10}$$

Consequently, Eq. (10) simultaneously represents the S-matrix of a Bargmann potential, phase-equivalent to the potential (1).

Let us next assume that the Bargmann potential Jost function $F(k)$ coincides with the Fredholm determinant $\mathcal{D}^{(+)}(k)$ of the potential (1):

$$F(k) = \frac{\prod\limits_{j=1}^{\mathcal{N}} (k + \mathrm{i}a_j)}{(k + \mathrm{i}b)^{\mathcal{N}}}. \tag{11}$$

The parameters $\{a_j\}$ must obey the following restrictions: either $a_j < 0$ corresponding to the introduction of new bound state, or $Re\, a_j > 0$. In the latter case the numerator in Eq. (11) should contain the factor $k + \mathrm{i}a_j^*$. So that in the absence of bound states the S-matrix has only pole in the upper half k plane, namely, in the point ib. This implies that, for instance, in the s-wave case the kernel $Q(r,\, r') = Q(t)$ $(t = r + r')$ of the Marchenko integral equation [4] is of the form

$$Q(t) = -\mathrm{i} \underset{k=\mathrm{i}b}{Res} \left\{ S(k)\, e^{\mathrm{i}kt} \right\}, \tag{12}$$

i. e. the separable kernel $Q(t)$ can be represented in the form of the sum of $\mathcal{N}$ components:

$$\begin{gathered}
Q(r,\, r') = \sum_{n=0}^{\mathcal{N}-1} Q_n^{(1)}(r)\, Q_n^{(2)}(r'), \\
Q_n^{(1)}(r) = r^n\, e^{-br}, \qquad Q_n^{(2)}(r) = e^{-br} \sum_{m=n}^{\mathcal{N}-1} A_m \binom{m}{n} r^{m-n}, \\
A_m = -\frac{(\mathrm{i})^{(m+1)}}{m!\,(\mathcal{N} - m - 1)!} \left\{ (k - \mathrm{i}b)^{\mathcal{N}} S(k) \right\}^{(\mathcal{N}-m-1)} \Bigg|_{k=\mathrm{i}b}.
\end{gathered} \tag{13}$$

3 Fredholm determinant in the J-matrix method

The general solution for multichannel Fredholm determinant for the model potential has been obtained in [10]. Here, we only express the single-channel Fredholm determinant in terms of the scattering inversion [6] results: the eigenvalues $\{\lambda_j\}$ and the elements $\{Z_{N,\,j}\}$ of the eigenvectors orthogonal matrix $\mathbf{Z}$ of the $N \times N$ Hamiltonian $\widehat{h} = \widehat{h}_0 + \frac{2\mu}{\hbar^2} \widehat{V}^\ell$ matrix calculated in the orthonormalized basis $\left\{ \left| \psi_n^\ell \right\rangle,\; n = \overline{0,\; N-1} \right\}$:

$$\left| \psi_n^\ell \right\rangle = d_n\, (2br)^{\ell+1}\, e^{-br}\, L_n^{2\ell+2}(2br), \qquad d_n = \sqrt{\frac{2bn!}{(n + 2\ell + 2)!}}. \tag{14}$$

For this purpose the solution for the S-matrix in the framework of the J-matrix method [10]:

$$S(k) = \frac{C_{N-1}^{\ell(-)}(k) - \mathcal{P}_N(k^2) J_{N-1,\,N}(k)\, C_N^{\ell(-)}(k)}{C_{N-1}^{\ell(+)}(k) - \mathcal{P}_N(k^2) J_{N-1,\,N}(k)\, C_N^{\ell(+)}(k)} \tag{15}$$

is used. Here, $J_{n,\,n'}(k)$ are the elements of the tridiagonal J-matrix: [1, 10]

$$\begin{gathered}
J_{n,\,n}(k) = \frac{1}{b}(n + \ell + 1) \frac{(n + 2\ell + 1)!}{n!} (b^2 - k^2), \\
J_{n,\,n+1}(k) = J_{n+1,\,n}(k) = \frac{(n + 2\ell + 2)!}{2\,b\,n!} (b^2 + k^2);
\end{gathered} \tag{16}$$

The $\mathcal{P}$-matrix $\mathcal{P}_N(k^2)$ is defined by [11, 6]

$$\mathcal{P}_N(k^2) = d_{N-1}^2 \sum_{j=1}^{N} \frac{Z_{N,\,j}^2}{k^2 - \lambda_j}. \tag{17}$$

Using the expression of so-called Casoratian determinant [11, 10]:

$$C^{\ell(+)}_{n+1}(k)\, S^{\ell}_{n}(k) - C^{\ell(+)}_{n}(k)\, S^{\ell}_{n+1}(k) = \frac{k}{J_{n,\,n+1}(k)}, \tag{18}$$

in view of three-term recursion relations [1]:

$$\begin{gathered}
J_{n,\,n-1}(k)\, S^{\ell}_{n-1}(k) + J_{n,\,n}(k)\, S^{\ell}_{n}(k) + J_{n,\,n+1}(k)\, S^{\ell}_{n+1}(k) = 0,\ n > 0, \\
J_{0,\,0}(k)\, S^{\ell}_{0}(k) + J_{0,\,1}(k)\, S^{\ell}_{1}(k) = 0, \\
J_{n,\,n-1}(k)\, C^{\ell(+)}_{n-1}(k) + J_{n,\,n}(k)\, C^{\ell(+)}_{n}(k) + J_{n,\,n+1}(k)\, C^{\ell(+)}_{n+1}(k) = 0,\ n > 0, \\
J_{0,\,0}(k)\, C^{\ell(+)}_{0}(k) + J_{0,\,1}(k)\, C^{\ell(+)}_{1}(k) = \frac{k}{S^{\ell}_{0}(k)},
\end{gathered} \tag{19}$$

the relation between the Fredholm determinant $\mathcal{D}^{(+)}(k)$ and the denominator of the expression (15) of the S-matrix can be developed:

$$\mathcal{D}^{(+)}(k) = -\frac{S^{\ell}_{0}(k)}{k}\, J_{N-1,\,N}(k) \left\{ C^{\ell(+)}_{N-1}(k) - \mathcal{P}_N(k^2) J_{N-1,\,N}(k)\, C^{\ell(+)}_{N}(k) \right\} \prod_{j=1}^{N} \frac{(k^2-\lambda_j)}{d^2_{j-1}\, J_{j-1,\,j}(k)}, \tag{20}$$

Using equations (7), (14) and (16) for the solutions $S^{\ell}_{n}(k)$, $C^{\ell(+)}_{n}(k)$, the normalization factors d_n and the elements $J_{n,\,n+1}(k)$, respectively, the expression (20) can be rewritten as

$$\begin{aligned}
\mathcal{D}^{(+)}(k) = {} & \frac{1}{(k+\mathrm{i}b)^{2N}}\, \frac{\ell!\,(N+2\ell+1)!}{(2\ell+1)!\,(N+\ell+1)!} \\
& \Bigg\{ (N+\ell+1)\, {}_2F_1 \left(-\ell,\ N\,;\, N+\ell+1\,;\, \left(\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)^2 \right) \prod_{j=1}^{N} (k^2-\lambda_j) - \\
& -N\,(k-\mathrm{i}b)^2\, {}_2F_1 \left(-\ell,\ N+1\,;\, N+\ell+2\,;\, \left(\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)^2 \right) \sum_{j=1}^{N} \left[Z^2_{N,\,j} \prod_{\substack{i=1 \\ i\neq j}}^{N} (k^2-\lambda_i) \right] \Bigg\}.
\end{aligned} \tag{21}$$

Since hypergeometric functions ${}_2F_1$ in the right-hand side of equation (21) are polynomials of degree ℓ in $\left(\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)^2$, $\mathcal{D}^{(+)}(k)$ represents a rational polynomials in k of degree $\mathcal{N}$:

$$\mathcal{D}^{(+)}(k) = \frac{\mathcal{R}_{\mathcal{N}}(k)}{(k+\mathrm{i}b)^{\mathcal{N}}}. \tag{22}$$

Notice that the leading coefficient of the polynomial $\mathcal{R}_{\mathcal{N}}(k)$ is, obviously, equal to

$$\begin{aligned}
\frac{\ell!\,(N+2\ell+1)!}{(2\ell+1)!\,(N+\ell+1)!} & \Big\{ (N+\ell+1)\, {}_2F_1\,(-\ell,\ N\,;\, N+\ell+1\,;\, 1) \\
& -N\, {}_2F_1\,(-\ell,\ N+1\,;\, N+\ell+2\,;\, 1) \sum_{j=1}^{N} Z^2_{N,\,j} \Big\} = 1,
\end{aligned} \tag{23}$$

since ${}_2F_1\,(a,\ b\,;\, c\,;\, 1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, and from the orthogonality of $\mathbf{Z}$ follows $\sum_{j=1}^{N} Z^2_{N,\,j} = 1$. Thus equation (8) is proved.

4 Examples

As a first example, the Bargmann potential has been obtained, phase-equivalent to the np $^1S_0$ Yamaguchi potential [7]:

$$V(k,k') = \Lambda_0 \frac{1}{(k^2+b^2)} \frac{1}{(k'^2+b^2)}, \tag{24}$$

where $\Lambda_0 = -76,4294\ MeV \cdot fm^{-1}$, $b = 1,158023\ fm^{-1}$. The potential (24) corresponds to the potential (1) of the rank $N = 1$ where $V_{0,0} = \frac{2\mu}{\hbar^2}\frac{\pi}{8b^2}\Lambda_0$. The Jost function (11) parameters b, $\{a_j\}$ are listed in Table. The corresponding Bargmann potential is displayed in Fig. 1.

In a second example the potential (1) of rank $N = 4$ has been constructed by the inverse scattering method [6] within the J-matrix approach. In doing this a set of Nijmegen [12] phase shifts in the $0-350\ MeV$ energy range is taken as input. The scaling parameter b and the set $\left\{\lambda_i,\ Z_{4,i}\,,\ i=\overline{1,\ 4}\right\}$ are given in Table determine the $N\times N$ matrix of the potential (1). $\mathcal{N} = 8$ parameters $\{a_j\}$ of the Bargmann potential Jost function (11) are listed in Table. Structure of deep and shallow parts of the Bargmann potential is shown in Figs. 2a-2d.

5 Conclusion

In summary, a neutral particle scattering from nonlocal separable potential with the Laguerre form factors has been dealt with in this work. For the potential (1) of rank N in the partial wave ℓ the Fredholm determinant was demonstrated to represent a rational polynomial in k of degree $\mathcal{N} = 2(N+\ell)$. Thus a correspondence between the potential (1) and the Bargmann potential may be established by identifying the Jost function of the latter with the Fredholm determinant of the model nonlocal potential. Here, the phase-equivalence between the potential (1) and its Bargmann's analogue is exact. As one goes to the Coulomb reference Hamiltonian $\widehat{h}_0$ or to the harmonic oscillator form factors case the Eq. (20) remains valid. However, the solutions $S_n^\ell(k)$, $C_n^{\ell(+)}(k)$ are not longer rational functions of the wave number k [10]. In this case Bargmann potential is only approximately phase-equivalent to the initial separable potential, since here the Fredholm determinant is approximated by rational expression (8). The parameters $\{a_j\}$ coincide with $\mathcal{N}$ (multiplied by imaginary unity) roots of the expression enclosed in figured parentheses in the Eq. (20).

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Table. The parameters values of the $np\ {}^1S_0$ Bargmann potentials, phase-equivalent to: (i) the Yamaguchi potential (24) of rank 1 [7]; (ii) the potential (1) of rank 4 from inverse scattering in the J-matrix approach. The parameters set $\left\{\lambda_i,\ \ Z_{4,\,i}\,,\ i=\overline{1,\ 4}\right\}$ of the separable potential is also presented here.

$b = 1.158023\,fm^{-1}$		$b = 1.3\,fm^{-1},\ N=4$				
j	a_j (fm^{-1})	i	$Z_{N,\,i}$	λ_i (fm^{-2})	j	a_j (fm^{-1})
1	2.276012669	1	.1493428930	.07258480091	1	$3.552401289 + 7.346450796\,\mathrm{i}$
2	.040033331	2	.4054072736	.6661380111	2	$3.552401289 - 7.346450796\,\mathrm{i}$
		3	.6619688746	3.203427534	3	$.8278500631 + 1.088427930\,\mathrm{i}$
		4	.6124857973	37.	4	$.8278500631 - 1.088427930\,\mathrm{i}$
					5	$.5554972974 + .5089227378\,\mathrm{i}$
					6	$.5554972974 - .5089227378\,\mathrm{i}$
					7	.4847238917
					8	.04377880951

Figure captions

Figure 1

The Bargmann *np* potential for $^1S_0$ channel, phase equivalent to the Yamaguchi potential [7].

Figure 2a

The Bargmann *np* potential for $^1S_0$ channel, phase equivalent to the potential (1) obtained by inverse scattering method in the J-matrix approach [6].

Figure 2b

Same as Fig. 2a. The structure of second deep well of the *np* Bargmann potential.

Figure 2c

Shallow structure of the *np* Bargmann potential shown in Fig. 2a-b.

Figure 2d

Same as Fig. 2c and asymptotic behaviour of the *np* Bargmann potential.

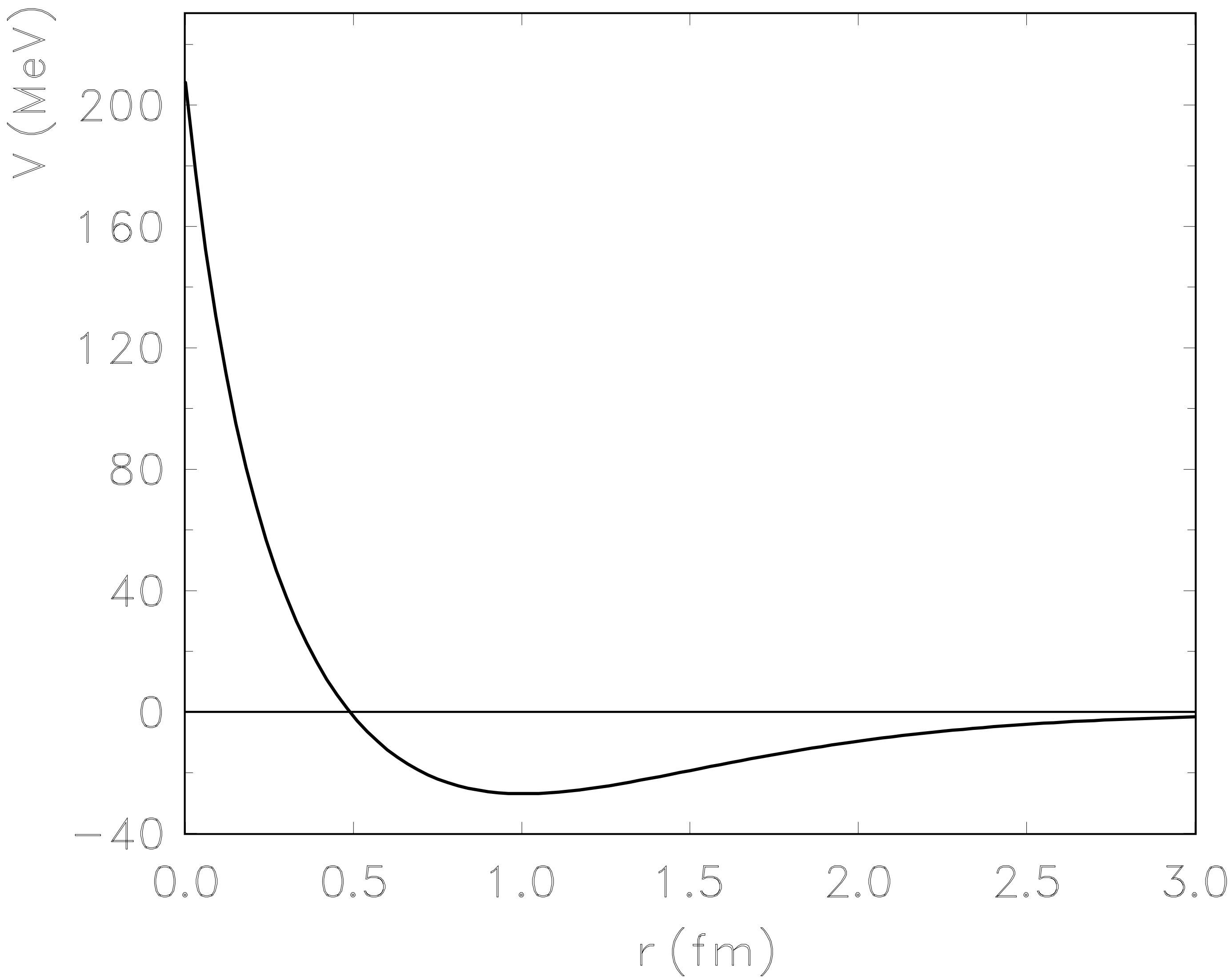


Figure 1.

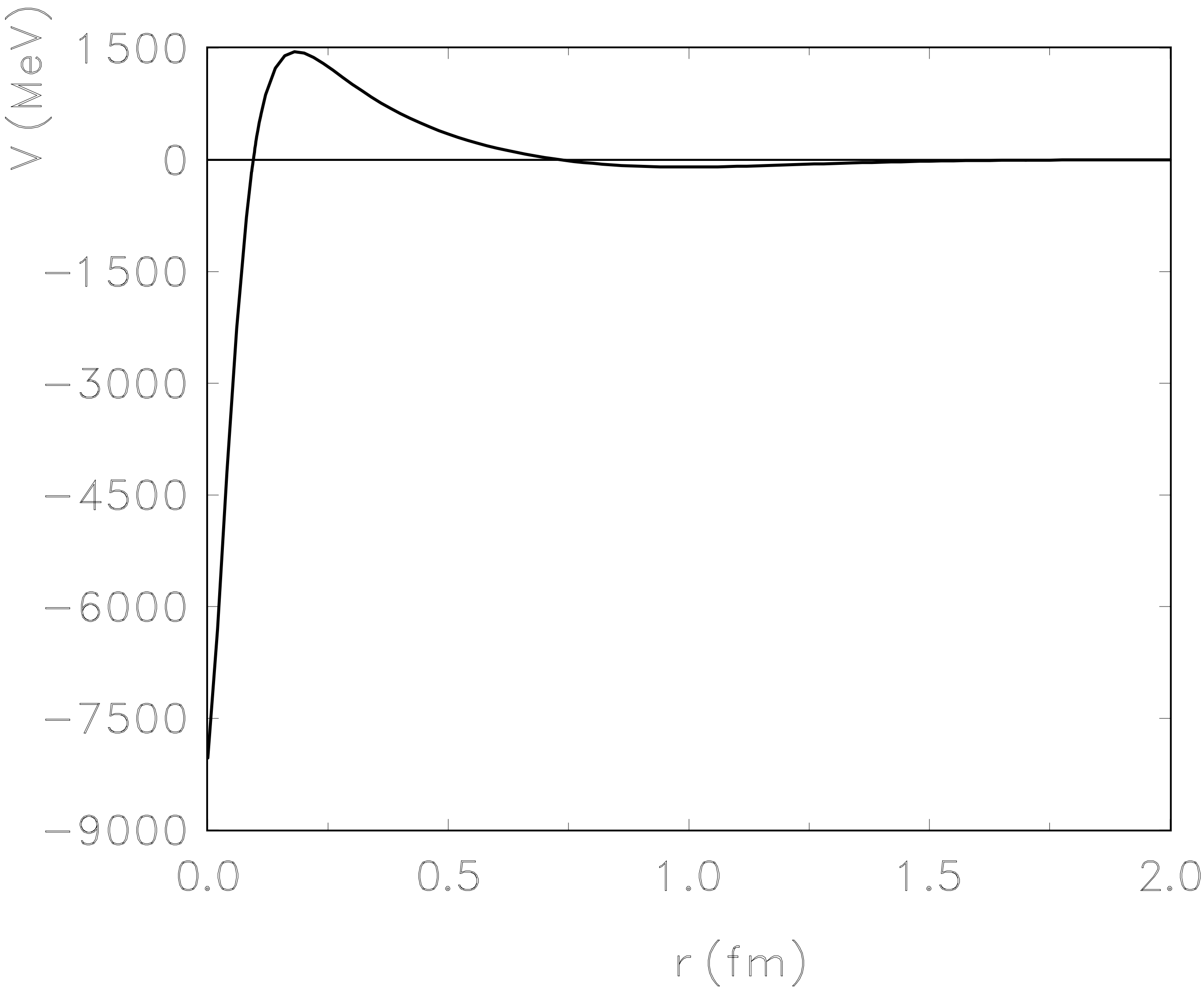
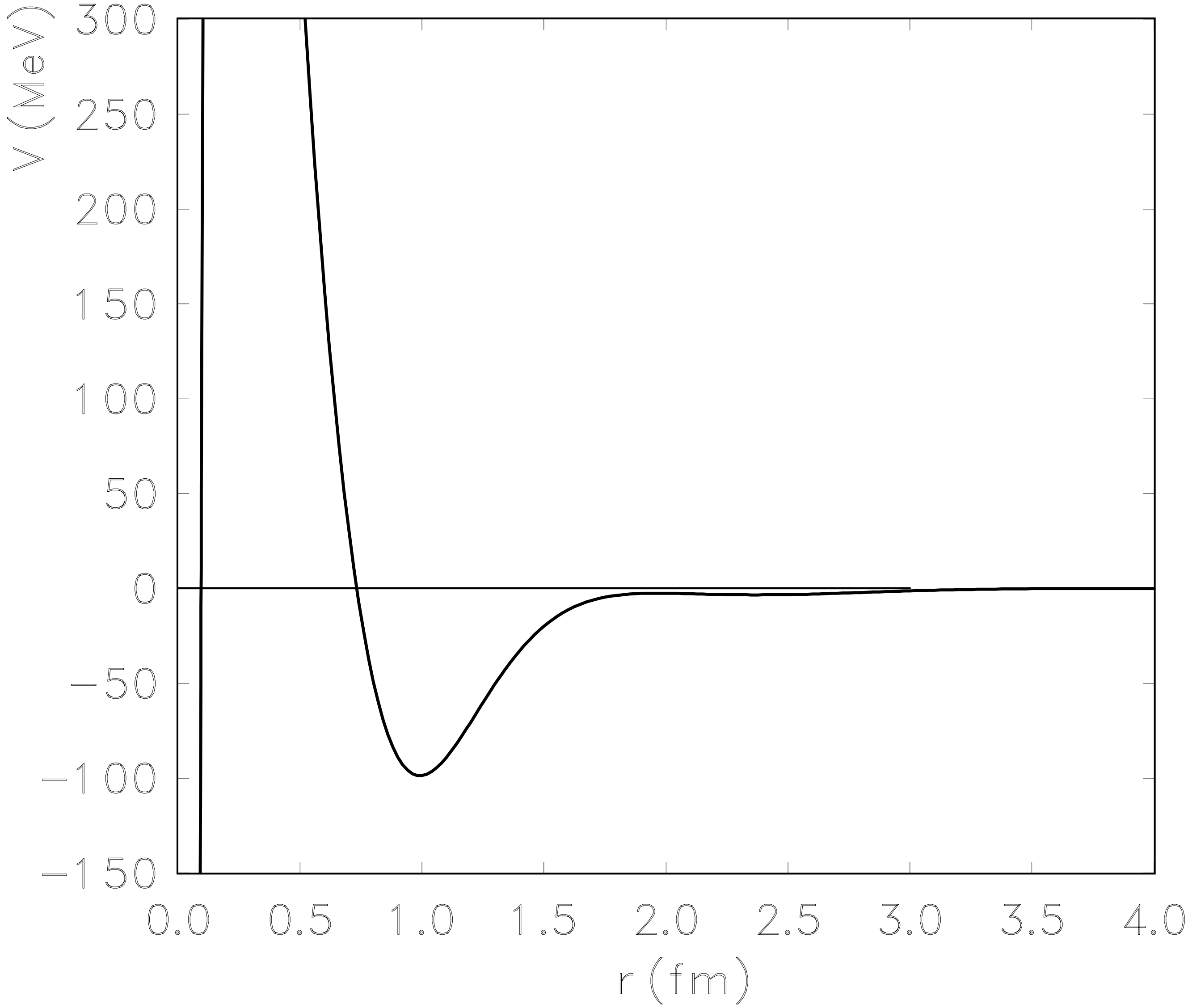


Figure 2a.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100
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−100
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2.0
2.5
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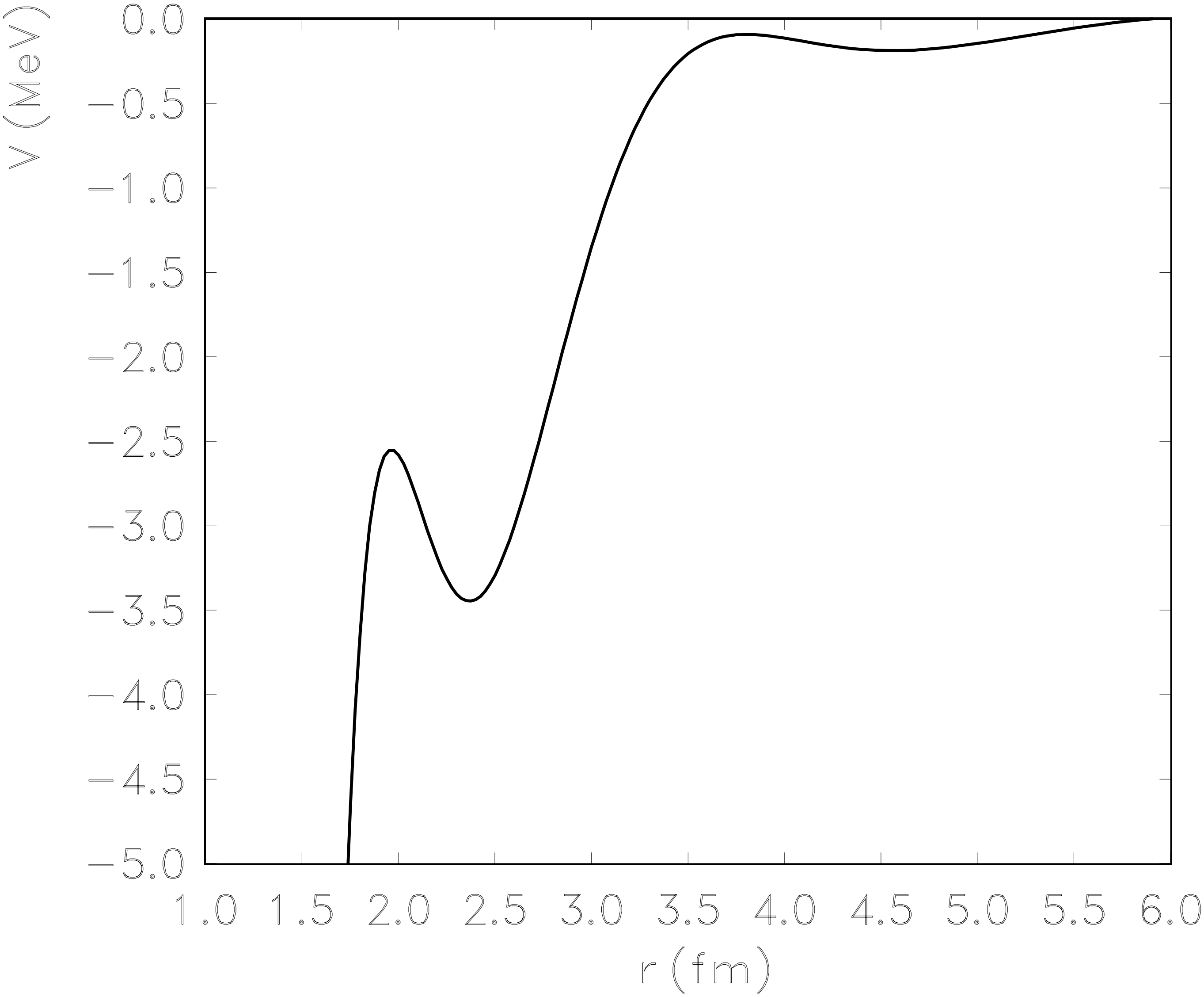
3.5
4.0
r (fm)

Figure 2b.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r (fm)

Figure 2c.

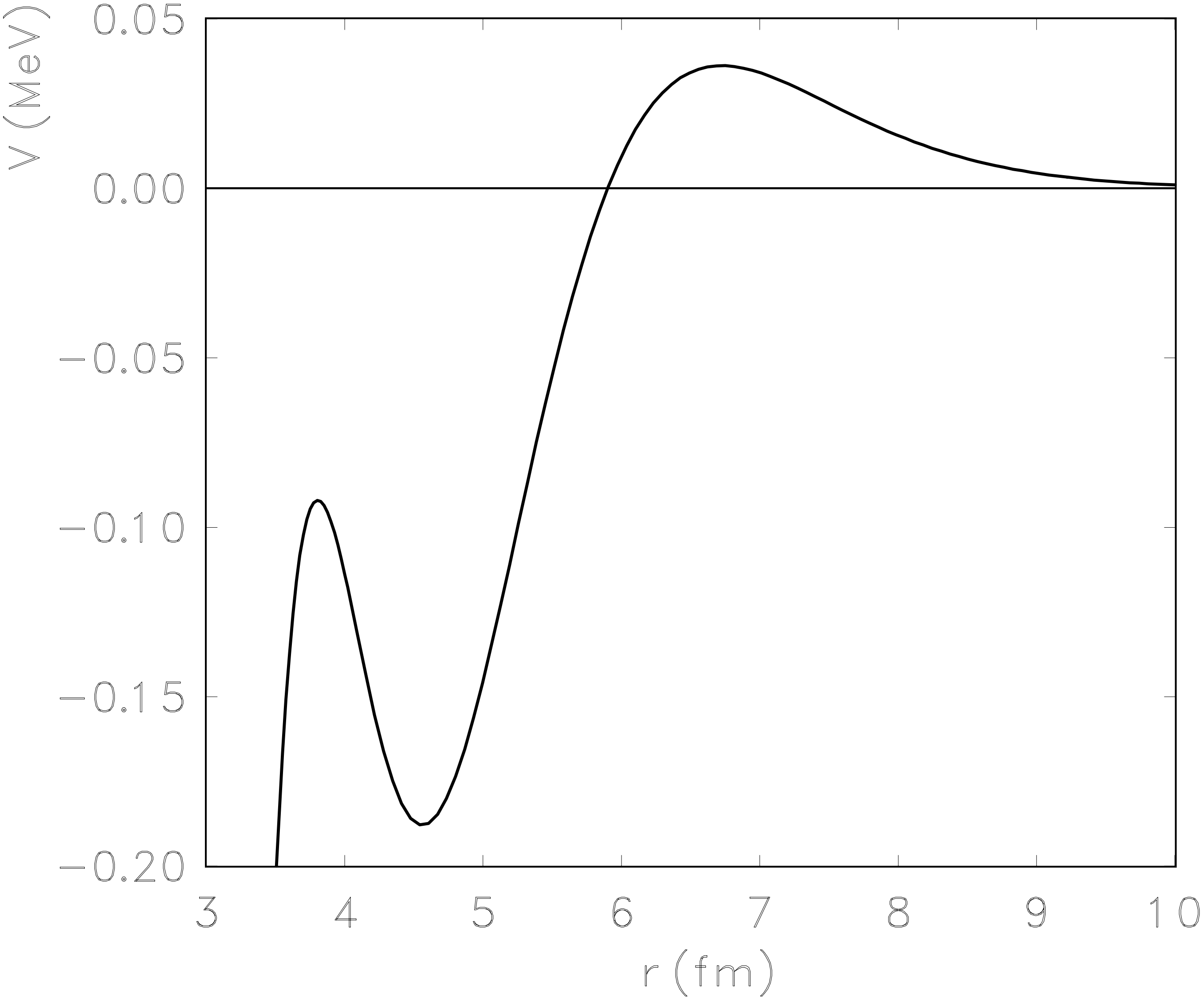

Figure 2d.